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Relativistic mechanics on rotating disk

In this paper we find equations of particle motion from the point of view of observer on a rotating disk. In [1] we found a local transformation from the inertial reference system to the noninertial rotating reference system, which generalizes the Lorentz transfromation. This transformation can be written in the following form with respect to cylindrical coordinates:

$$dr' = dr, \quad r'd\phi' = \frac{r'd\phi - \dfrac{\omega r dt}{c^2}}{\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}}, \quad dt' = \frac{dt - \dfrac{\omega r}{c^2} r d\phi}{\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}} \tag{1}$$

Moreover, in [1] we have proved that the transformation (1) preserves the Lorentz metric,

$$ds^2 = -dr^2 - r^2 d\phi^2 + c^2 dt^2 = -dr'^2 - r'^2 d\phi'^2 + c^2 dt'^2. \tag{2}$$

The transformation (1) is nonholonomic, i.e. the transformation matrix

$$||A_i^{i'}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}} & \dfrac{-\omega}{\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}} \\ 0 & \dfrac{-\omega r^2}{c^2\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}} & \dfrac{1}{\sqrt{1-\dfrac{\omega^2 r^2}{c^2}}} \end{pmatrix}$$

is not a Jacobian matrix of any coordinate transformation. This is important when we are constructing mechanics on rotaing disk. We suppose that with respect to the noninertial reference system the particle motion is subordinate to the the law

$$\frac{\delta v^{i'}}{\delta s} \equiv \frac{dv^{i'}}{ds'} + \Gamma^{i'}_{j'k'} \frac{dx^{j'}}{ds'} \frac{dx^k}{ds'} = F^{i'}/m_0 \qquad (3),$$

where m_0 is the rest mass of particle, $v^{i'} \equiv \dfrac{dx^{i'}}{ds}$, $F^{i'}$ is the effective force in the noninertial reference system, $\Gamma^{i'}_{j'k'}$ are the connection coefficients with respect to the noninertial reference system, which are obtained from the Christoffel symbols $\Gamma^i i_{jk}$ of the metric (2)

in the following way:

$$\Gamma^{i'}_{j'k'} = A^{i'}_{k} A^{j}_{j'} \frac{\partial A^{k}_{k'}}{\partial x^{j}} + \Gamma^{i}_{jk} A^{i'}_{i} A^{k}_{k'} A^{j}_{j'}. \tag{4}$$

After easy calculations we get

$$\begin{aligned}
&\Gamma^{0'}_{0'1'} = -\frac{\omega^2 r/c^2}{1 - \frac{\omega^2 r^2}{c^2}}; \quad \Gamma^{0'}_{1'2'} = -\Gamma^{0'}_{2'1'} = \frac{\omega r/c^2}{(1 - \frac{\omega^2 r^2}{c^2})}; \\
&\Gamma^{1'}_{0'0'} = -\frac{r\omega^2}{1 - \frac{\omega^2 r^2}{c^2}}; \quad \Gamma^{1'}_{0'2'} = \Gamma^{1'}_{2'0'} = -\frac{r\omega}{1 - \frac{\omega^2 r^2}{c^2}}; \quad \Gamma^{1'}_{2'2'} = -\frac{r}{1 - \frac{\omega^2 r^2}{c^2}}; \\
&\Gamma^{2'}_{0'1'} = \Gamma^{2'}_{1'0'} = \frac{\omega}{r(1 - \frac{r\omega^2 r^2}{c^2})}; \quad \Gamma^{2'}_{1'2'} = \frac{1}{r}; \quad \Gamma^{2'}_{2'1'} = \frac{1}{r(1 - \frac{\omega^2 r^2)}{c^2}};
\end{aligned} \tag{5}$$

The other connection coefficients are zero.

Now let us write the motion equation for a particle with nonzero rest mass ($ds^2 \neq 0$) and for a particle with zero rest mass ($ds^2 = 0$) , with respect to the noninertial reference system on rotating disk.

Using (5), we get for the case $ds^2 \neq 0$

$$\begin{aligned}
&\frac{d^2 t}{ds^2} - \frac{\omega^2 r/c^2}{1 - \frac{\omega^2 r^2}{c^2}} \frac{dt}{ds} \frac{dr}{ds} = \frac{1}{m} F^0; \\
&\frac{d^2 r}{ds^2} - \frac{r\omega^2}{1 - \frac{\omega^2 r^2}{c^2}} (\frac{dt}{ds})^2 - \frac{2\omega r}{1 - \frac{\omega^2 r^2}{c^2}} \frac{dt}{ds} \frac{d\phi}{ds} - \frac{r}{1 - \frac{\omega^2 r^2}{c^2}} (\frac{d\phi}{ds})^2 = \frac{1}{m} F^1; \\
&\frac{d^2 \phi}{ds^2} = \frac{2\omega}{r(1 - \frac{\omega^2 r^2}{c^2})} \frac{dt}{ds} \frac{dr}{ds} + \frac{2 - \frac{\omega^2 r^2}{c^2}}{r(1 - \frac{\omega^2 r^2}{c^2})} \frac{dr}{ds} \frac{d\phi}{ds} = \frac{1}{m} F^2
\end{aligned}$$

For a photon ($ds^2 = 0$), we set $\frac{dx^i}{d\sigma} = K^i$, where σ is the canonical parameter, and K^i are components of the wave vector. We get

$$\frac{dK^0}{d\sigma} - \frac{\omega^2 r/c^2}{1 - \dfrac{\omega^2 r^2}{c^2}} K^0 K^1 = F^0;$$

$$\frac{dK^1}{d\sigma} - \frac{\omega^2 r}{1 - \dfrac{\omega^2 r^2}{c^2}} (K^0)^2 - \frac{2\omega r}{1 - \dfrac{\omega^2 r^2}{c^2}} K^0 K^2 - \frac{r}{1 - \dfrac{\omega^2 r^2}{c^2}} (K^2)^2 = F^1;$$

$$\frac{dK^2}{d\sigma} = \frac{2\omega}{r(1 - \dfrac{\omega^2 r^2}{c^2}} K^0 K^1 + \frac{2 - \dfrac{\omega^2 r^2}{c^2}}{r(1 - \dfrac{\omega^2 r^2}{c^2})} K^1 K^2 = F^2$$

Thus we see that a particle moving along a rotating disk is influenced by forces arising from geometry, besides the external forces F^i. These forces connected with nonholonomity of reference system are generated by the noninertiality. They can be considered as analogs of the centrifugal forces and the Coriolis forces.

References

1. V.Bashkov, M. Malakhaltsev *Geometry of rotating disk and Sagnac effect*, arXiv: gr-qc/0011061, 18 Nov. 2000.

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